Propagation of TE waves in cylindrical nonlinear dielectric waveguides

H. W. Schürmann,[1] Y. Smirnov,[2] and Y. Shestopalov[3]
[1]*Department of Physics, University of Osnabrück, Barbarastrasse 7, D-49069 Osnabrück, Germany*
[2]*Department of Mathematics and Mathematical Modeling, Penza State University, ul. Krasnaya, 40, 440017 Penza, Russia*
[3]*Institute of Engineering Sciences, Physics, and Mathematics, Karlstad University, SE-65188 Karlstad, Sweden*
(Received 11 June 2004; published 25 January 2005)

The propagation of TE-polarized electromagnetic waves along a Kerr-type nonlinear dielectric, nonabsorbing, nonmagnetic, and isotropic (circular) cylindrical waveguide is investigated. For axially (azimuthal) symmetric solutions the problem is reduced to a cubic-nonlinear integral equation that is solved by iteration leading to a sequence uniformly convergent to the solution of the integral equation. The dispersion relations associated to the exact and iterate solutions, respectively, are derived and solved, subject to certain constraints. The roots of the exact dispersion relation are approximated by the roots of the dispersion relations generated by the iterate solutions. All statements of existence and convergence are based on results of a previous paper. Numerical results (concerning solutions of dispersion relations, field patterns, dependence of the propagation constant and of the cutoff radius on the nonlinearity parameter, power flow) are included.

DOI: 10.1103/PhysRevE.71.016614 PACS number(s): 42.65.Tg

I. INTRODUCTION

The propagation of electromagnetic waves in linear media along a circular cylindrical dielectric waveguide is a relevant topic of classical electrodynamics [1,2]. Cylindrical waveguide structures consisting of nonlinear media have been investigated by several authors. Chiao *et al.* [3] considered self-trapping of optical beams and calculated the radial profile of the electric field. Eleonskii *et al.* [4] presented a theory of cylindrical self-focusing waveguides primarily based on the qualitative analysis of phase trajectories corresponding to solutions of Maxwell's equations including numerical calculations of the fundamental and lowest nonfundamental TE mode. Based on a variational technique Chen [5] derived analytical solutions for the fundamental mode of nonlinear fibers with arbitrary nonlinearity. Akhmediev *et al.* [6] studied the modulation instability of the fundamental mode of a Kerr-law nonlinear cylindric waveguide and presented analytical and numerical results by using certain integral relations. Sammut and Pask [7] used a variational formulation of the wave equation for waveguides with arbitrary nonlinearity of the optical fiber and presented analytical approximations for the field profiles. Recently, Sjoberg [8] analyzed the propagation of electromagnetic waves in a nonlinear cylindrical waveguide by means of a perturbation approach with the strength of the nonlinearity as the perturbation parameter.

In the case of three-layer planar waveguide with Kerr nonlinearity the fields and the dispersion relation can be expressed exactly in terms of elliptic functions [9]. For the corresponding Kerr-law nonlinear dielectric cylindrical waveguide some mathematical results have been presented in a previous paper [10].

The present paper draws some physical conclusions from Ref. [10]. Maxwell's equations are reduced to a nonlinear integral equation with a kernel in the form of a Green's function of the Bessel equation. Based on the existence of a unique and continuous solution of the integral equation and the uniform convergence of the sequence of iterate solutions [11], an approximate analytical solution is presented and compared with the exact numerical solution.

Propagating TE waves are associated with the existence of propagation constants as solutions of the dispersion relation. Due to the existence of solutions of the dispersion relations generated by the exact and the iterate solutions of the integral equation, respectively [12], an approximate dispersion relation is presented and evaluated to yield expressions for field patterns, dispersion curves, cutoff radii, and power flow.

II. FORMULATION OF THE PROBLEM

We consider the wave propagation in a cylindrical lossless dielectric waveguide with the circular cross section $W=\{(x,y):\rho=\sqrt{x^2+y^2}<R\}$. The waveguide is assumed to be isotropic and homogeneous in z direction (the waveguide axis). The (real) electric field (in cylindrical coordinates)

$$\vec{E}(\rho,\varphi,z,t)=\vec{E}_+(\rho,\varphi,z)\cos\omega t+\vec{E}_-(\rho,\varphi,z)\sin\omega t \quad (2.1)$$

satisfies

$$\text{rotrot}\,\vec{\mathcal{E}}(\rho,\varphi,z)=\omega^2\varepsilon\mu\vec{\mathcal{E}}(\rho,\varphi,z), \quad (2.2)$$

where

$$\vec{\mathcal{E}}(\rho,\varphi,z)=\vec{E}_+(\rho,\varphi,z)+i\vec{E}_-(\rho,\varphi,z). \quad (2.3)$$

The permittivity ε of the internal medium is assumed to have a Kerr-nonlinear dependence on the electric field intensity according to

$$\varepsilon=\begin{cases}\varepsilon_2+a|\vec{\mathcal{E}}|^2, & 0\leqslant\rho\leqslant R,\\ \varepsilon_1=\text{const}, & \rho>R,\end{cases} \quad (2.4)$$

where ε_1, ε_2, and a are real constants (with respect to ρ, φ, z, t). We further assume all media to be nonmagnetic with $\mu=\mu_0$ being the free-space permeability.

The field $\vec{\mathcal{E}}$ propagates along the waveguide z axis as

$$\vec{\mathcal{E}}(\rho,\varphi,z) = \vec{V}(\rho,\varphi)e^{i\gamma z}, \tag{2.5}$$

where γ is the propagation constant.

For a linear medium, the method to obtain solutions of Maxwell's equations in a cylindrical structure is well documented in the literature [1,2]. In particular, due to the boundary conditions, the types of fields do not, in general, separate into TE and TM modes, except in special circumstances such as azimuthal symmetry in circular cylinders, to be investigated below. Assuming a separable solution $\vec{V}_{E,H}(\rho,\varphi)$, harmonic functions $\sin n\varphi$, $\cos n\varphi$ ($n \geqslant 0$ integer) for the azimuthal dependence and, for the radial dependence, cylinder functions that fulfill a second order Bessel differential equation are obtained.

In analogy to the ansatz for planar waveguides [9] ansatz (2.5) models "stationary" solutions with a z-independent (in a lossless medium) amplitude V. For lossy media or if there are any (small) perturbations the z dependence of V must be taken into account, leading (by using the slowly varying envelope approximation) to a nonlinear Schrödinger equation for V; thus one can perform a stability analysis of the solutions found. As will be seen below these solutions are similar to those of the linear case. We do not consider the problem whether the solutions found are stable (and thus solitons).

In the problem under study we proved [10] the existence of TE waves. Omitting the factors $\sin n\varphi$ and $\cos n\varphi$ we consider the polarization case $\vec{\mathcal{E}}=\{0;\mathcal{E}_\varphi;0\}$, $\vec{\mathcal{H}}=\{\mathcal{H}_\rho;0;\mathcal{H}_z\}$ in the following. Then Maxwell's equations imply that $\mathcal{H}_\rho$ and $\mathcal{H}_z$ do not depend on φ and we obtain [10]

$$\mathcal{H}_\rho = -\frac{k_0}{i\omega\mu}\frac{\partial \mathcal{E}_\varphi}{\partial z}, \quad \mathcal{H}_z = \frac{k_0}{i\omega\mu}\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\mathcal{E}_\varphi), \tag{2.6}$$

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\mathcal{E}_\varphi)\right) + \frac{\partial^2\mathcal{E}_\varphi}{\partial z^2} + \omega^2\varepsilon\mu\mathcal{E}_\varphi = 0. \tag{2.7}$$

According to Eq. (2.5) we assume $\mathcal{E}_\varphi(\rho,z)=E_0 u(\rho,\gamma^2)e^{i\gamma z}$ (with E_0 a real constant) so that Eq. (2.7) can be written as

$$\left(\frac{1}{\rho}(\rho u)'\right)' + (\omega^2\varepsilon\mu - \gamma^2)u = 0, \tag{2.8}$$

where the prime denotes the differentiation with respect to ρ. Since $\varepsilon=\varepsilon_1$ outside the waveguide, we obtain the Bessel equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u - k_1^2 u = 0, \quad \rho > R, \tag{2.9}$$

where $k_1^2=\gamma^2-\omega^2\varepsilon_1\mu_0$.

Inside the waveguide, where $\varepsilon=\varepsilon_2+a|\vec{\mathcal{E}}|^2$, we obtain a cubic-nonlinear second-order differential equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_2^2 u + \alpha u^3 = 0, \quad 0 < \rho < R, \tag{2.10}$$

where $\alpha=\omega^2\mu_0 a E_0^2 = k_0^2 a E_0^2/\varepsilon_0$, $k_0^2=\omega^2\varepsilon_0\mu_0$, and $k_2^2=\omega^2\varepsilon_2\mu_0 - \gamma^2$. The continuity of $\mathcal{E}_\varphi$ and $\mathcal{H}_z$ at the interface $\rho=R$ leads to the conditions

$$u(R+0) = u(R-0), \quad u'(R-0) = u'(R+0). \tag{2.11}$$

The problem is (i) to find nonvanishing functions $u(\rho,\gamma^2)$ bounded at $\rho=0$ and continuously differentiable on the semi-infinite interval $\rho>0$ and (ii) to find triples $\{R,\gamma,\alpha\}$ such that $u(\rho,\gamma^2)$ satisfies Eqs. (2.9) and (2.10), and the boundary conditions (2.11).

III. SOLUTIONS AND DISPERSION RELATION

In order to fulfill the radiation condition at infinity, we choose the solution of Eq. (2.9) in the form

$$u = C_1 K_1(k_1\rho), \quad \rho > R, \tag{3.1}$$

where K_1 is the Macdonald function (Hankel function for pure imaginary arguments). Normalizing the field according to $\mathcal{E}_\varphi(R,0) \equiv E_0$, we obtain

$$u = \frac{K_1(k_1\rho)}{K_1(k_1 R)}, \quad \rho > R. \tag{3.2}$$

The radiation condition is fulfilled because $K_1(k_\rho) \to 0$ exponentially as $\rho \to \infty$.

We turn to the solution of Eq. (2.10) and write it in the form

$$Lu + \alpha\rho u^3 = 0, \quad L = \rho\frac{d^2}{d\rho^2} + \frac{d}{d\rho} + \left(k_2^2\rho - \frac{1}{\rho}\right). \tag{3.3}$$

Using standard methods [13], Green's function G for the mixed-type boundary value problem

$$LG = -\delta(\rho - s), \quad G|_{\rho=0} = G'|_{\rho=R} = 0 \quad (0 < s < R) \tag{3.4}$$

is given by (J_1 and N_1 denote the Bessel function and the Neumann function of first kind, respectively)

$$G(\rho,s) = \begin{cases} \dfrac{\pi}{2}\left[\dfrac{J_1(k_2\rho)J_1(k_2 s)}{J_1'(k_2R)}N_1'(k_2R) - J_1(k_2\rho)N_1(k_2 s)\right], & \rho < s \leqslant R, \\ \dfrac{\pi}{2}\left[\dfrac{J_1(k_2\rho)J_1(k_2 s)}{J_1'(k_2R)}N_1'(k_2R) - J_1(k_2 s)N_1(k_2\rho)\right], & s < \rho \leqslant R. \end{cases} \tag{3.5}$$

Applying the second Green's formula

$$\int_0^R (vLu - uLv)d\rho = \int_0^R [v(\rho u')' - u(\rho v')']d\rho$$
$$= R[u'(R)v(R) - v'(R)u(R)] \quad (3.6)$$

and setting $v \equiv G$, we obtain, using Eq. (3.4),

$$\int_0^R (GLu - uLG)d\rho = R[u'(R-0)G(R,s) - G'(R,s)u(R-0)]$$
$$= Ru'(R-0)G(R,s). \quad (3.7)$$

The left-hand side of Eq. (3.7) can be expressed by means of Eq. (3.3). Hence

$$\int_0^R (GLu - uLG)d\rho = -\alpha\int_0^R G\rho u^3 d\rho + u(s,\gamma^2), \quad (3.8)$$

leading to an integral representation of the solution to Eq. (2.10)

$$u(s,\gamma^2) = \alpha\int_0^R G(\rho,\rho_0)\rho u^3(\rho)d\rho + Ru'(R+0)G(R,\rho_0),$$
$$0 \leqslant \rho_0 \leqslant R, \quad (3.9)$$

where the condition $u'(R-0)=u'(R+0)$ has been used.

By means of well-known relations between Bessel functions [14] the Green's function $G(R,\rho_0)$ in Eq. (3.9) can be expressed by $G(R,\rho_0)=(1/k_2R)[J_1(k_2\rho_0)/J_1'(k_2R)]$. Thus the solution of Eq. (2.10) satisfies the nonlinear integral equation

$$u(s,\gamma^2) = \alpha\int_0^R G(\rho,s)\rho u^3(\rho)d\rho + f(s,\gamma^2), \quad 0 \leqslant s \leqslant R, \quad (3.10)$$

with, taking advantage of the continuity condition (2.11),

$$f(s,\gamma^2) = \frac{k_1K_1'(k_1R)J_1(k_2s)}{k_2K_1(k_1R)J_1'(k_2R)}. \quad (3.11)$$

The unique solution $u(s,\gamma^2)$ of Eq. (3.10) exists [15] if

$$|\alpha| < A^2, \quad (3.12)$$

where

$$A(R,\gamma^2) = \frac{2}{3\sqrt{3}}\frac{1}{\max\limits_{s\in[0,R]}|f(s)|\sqrt{\max\limits_{s\in[0,R]}\int_0^R d\rho|\rho G(\rho,s)|}} \quad (3.13)$$

[in the notation of $A(R,\gamma^2)$ the implicit dependence of γ^2 on α according to Eq. (3.17) (below) has been suppressed].

The function $u(s,\gamma^2)$ is continuous with respect to s and γ^2 [16]. It can be approximated by means of the iteration procedure

$$u_{n+1}(s,\gamma^2) = \alpha\int_0^R d\rho\, \rho G(\rho,s)u_n^3(\rho,\gamma^2) + f(s,\gamma^2), \quad n=0,1,\ldots \quad (3.14)$$

in the sense that the iteration sequence (3.14) converges uniformly to the exact solution $u(s,\gamma^2)$ of Eq. (3.10) [17]; namely, $\max_{0\leqslant s\leqslant R}|u_{n+1}(s,\gamma^2)-u(s,\gamma^2)| \to 0$, $n\to\infty$.

Consistent with the normalization according to Eq. (3.2) it is useful to start with

$$u_0(s,\gamma^2) = \frac{J_1(k_2s)}{J_1(k_2R)}. \quad (3.15)$$

Applying the continuity condition $u(R-0)=u(R+0)$ to Eq. (3.10) we obtain *the dispersion relation* associated to the exact solution $u(s,\gamma^2)$

$$f(R,\gamma^2) + \alpha\int_0^R d\rho\, \rho G(\rho,R)u^3(\rho,\gamma^2) = 1, \quad (3.16)$$

that can be rewritten as

$$g(R,\gamma^2) = \alpha F(R,\gamma^2;u^3), \quad (3.17)$$

with

$$g(R,\gamma^2) = k_2RK_1(k_1R)J_1'(k_2R) - k_1RK_1'(k_1R)J_1(k_2R)$$
$$= k_2RK_1(k_1R)J_0(k_2R) + k_1RK_0(k_1R)J_1(k_2R), \quad (3.18)$$

$$F(R,\gamma^2;u^3) = K_1(k_1R)\int_0^R d\rho\, \rho J_1(k_2\rho)u^3(\rho,\gamma^2). \quad (3.19)$$

Subject to certain conditions, triples $\{R,\gamma^2,\alpha\}$ exist that satisfy the exact nonlinear dispersion relation (3.17) (see Ref. [10], Sec. V). Thus the problem stated in Sec. II has a nontrivial solution. We note that Eq. (3.17) is identical with the linear dispersion relation $g(R,\gamma^2)=0$ [18] if $\alpha=0$. In this case, $f(s,\gamma^2)$ given by Eq. (3.11) is equal to $u_0(s,\gamma^2)$ according to Eq. (3.15).

It seems useful to compare the foregoing approach with the "standard" approach (see [19]). According to Ref. [19], Eq. (2.3.14), the "standard" approach starts with a separation ansatz (with a z-dependent amplitude) leading to a system of two equations [cf. Eqs. (2.3.15) and (2.3.16) in Ref. [19]] solved using the first-order perturbation theory where the nonlinear dielectric function is replaced by a constant linear one. The result is a certain propagation constant β and the corresponding modal field distribution $F(\rho)$. The nonlinearity is taken into account by adding a correction term $\Delta\beta$ [determined by the unperturbed field $F(\rho)$], leaving $\underline{F}(\rho)$ unchanged. The dispersion relation is given by $\overline{\beta}=\beta+\Delta\beta$ within this approach [cf. Eq. (2.3.19) in Ref. [19]]. Obviously the ansatz of the "standard" approach is more general than ansatz (2.5). We have considered a particular polarization case, disregarding stability, and restricted analysis to lossless media with the core homogeneous in the z direction.

On the other hand, we think that our approach is more rigorous (mathematically). The field obeys the nonlinear integral equation (3.10); the exact solution to this equation exists and can be approximated by iteration and solving the (exact) dispersion relation (3.17) whose exact solution also exists and can be approximated for nonlinearities α subject to the conditions of the theorem presented in the Appendixes. It would be intriguing to compare practical results of both methods.

IV. APPROXIMATIONS

It is obvious and useful to consider the sequence of dispersion relations generated by the iteration sequence (3.14) (for α, R prescribed)

$$g(R,(\gamma^{(n)})^2) = \alpha F(R,(\gamma^{(n)})^2;u_n^3). \tag{4.1}$$

Here u_n is determined from Eq. (3.14) with $\gamma=\gamma^{(n-1)}$ and $\gamma^{(n)}$ from Eq. (3.17) in which u is replaced by u_n.

We have shown [20] that solutions $(\gamma^{(n)})^2$ to Eq. (4.1) exist and that the sequence $(\gamma^{(n)})^2$ approximates the solutions γ^2 of the exact dispersion relation (3.17) $[|(\gamma^{(n)})^2-\gamma^2|\to 0$ as $n\to\infty]$. In this meaning, the sequence of dispersion relations defined by Eq. (4.1) approximates the exact relation (3.17).

The first approximation is obtained by inserting $u_0(s,\gamma^2)$ into Eq. (3.17). Hence it follows

$$H(R,\gamma,\alpha) \equiv g(R,\gamma^2) - \alpha\frac{K_1(k_1R)}{J_1^3(k_2R)}\int_0^R d\rho\, \rho J_1^4(k_2\rho) = 0. \tag{4.2}$$

The solutions $(\gamma^{(0)})^2$ of Eq. (4.2) and the zeroth approximation $u_0(s,(\gamma^{(0)})^2)$ given by Eq. (3.15) must be inserted into Eq. (3.14) leading to the first approximation $u_1(s,(\gamma^{(0)})^2)$ of the field function $u(s,\gamma^2)$

$$\begin{aligned} u_1(s,(\gamma^{(0)})^2) = {} & \frac{k_1K_1'(k_1R)J_1(k_2s)}{k_2K_1(k_1R)J_1'(k_2R)} + \alpha\frac{\pi}{2}\Bigg\{\frac{N_1'(k_2R)}{J_1'(k_2R)}J_1(k_2s) \\ & \times\int_0^R d\rho\, \rho J_1(k_2\rho)u_0^3(\rho,(\gamma^{(0)})^2) - N_1(k_2s) \\ & \times\int_0^s d\rho\, \rho J_1(k_2\rho)u_0^3(\rho,(\gamma^{(0)})^2) - J_1(k_2s) \\ & \times\int_s^R d\rho\, \rho N_1(k_2\rho)u_0^3(\rho,(\gamma^{(0)})^2)\Bigg\}, \end{aligned} \tag{4.3}$$

where $k_1=\sqrt{(\gamma^{(0)})^2-\varepsilon_1}$ and $k_2=\sqrt{\varepsilon_2-(\gamma^{(0)})^2}$.

Since $\varepsilon_2-j_{1i}^2/R^2<(\gamma^{(0)})^2<\varepsilon_2-j_{0i}^2/R^2$ holds (cf. Appendix A) $J_1'(k_2R)$ cannot be zero because the zeros of J_1' are outside the interval $[\gamma_{1i}^2,\gamma_{2i}^2]$. The same statement is valid for the solutions γ^2 of Eq. (3.17), so that Green's function (3.5) exists and $u_1(s)$, $s\in[0,R]$, is nonsingular.

If $\alpha=0$, u_1 is equal to f and this is equal to $u_0(s,\gamma^2)$ because $(\gamma^{(0)})^2=\gamma^2$ is a solution of the linear dispersion relation $g(R,\gamma^2)=0$.

Summing up,

$$\tilde{u}(s) = \begin{cases} u_1(s,(\gamma^{(0)})^2), & 0\leqslant s\leqslant R, \\ K_1(k_1s)K_1^{-1}(k_1R), & s>R, \end{cases}$$

where $u_1(s,(\gamma^{(0)})^2)$, given by Eq. (4.3), is an approximate solution of the problem stated in Sec. II if $(\gamma^{(0)})^2$ is a solution of the approximate dispersion relation (4.2).

V. APPLICATIONS

We introduce the dimensionless variables and parameters $\tilde{\rho}=k_0\rho$, $\tilde{z}=k_0z$, $\tilde{R}=k_0R$, $\tilde{\varepsilon}=\varepsilon/\varepsilon_0$, $\tilde{k}_1=\sqrt{\tilde{\gamma}^2-\tilde{\varepsilon}_1}$, $\tilde{k}_2=\sqrt{\tilde{\varepsilon}_2-\tilde{\gamma}^2}$, $(\tilde{\varepsilon}_2>\tilde{\varepsilon}_1)$, and $\tilde{\gamma}=\gamma/k_0$. Thus the nonlinearity parameter α in Eqs. (4.2) and (4.3) is dimensionless and given by $\alpha=aE_0^2/\varepsilon_0$. In the following we omit the tildes.

To apply the results of the previous section it is useful to evaluate the quantities A given by Eq. (3.13) and A_1 and B_1 defined in Appendix A. According to Appendix B we obtain for the first interval $\gamma^2\in\Lambda_1$, $\Lambda_1=[\varepsilon_2-j_{11}^2/R^2,\varepsilon_2-j_{01}^2/R^2]$ (the first mode)

$$A(\gamma^2,R) = \left|\frac{0.228713(\gamma^2-\varepsilon_2)[J_0(R\sqrt{\varepsilon_2-\gamma^2})-J_2(R\sqrt{\varepsilon_2-\gamma^2})]K_1(R\sqrt{\gamma^2-\varepsilon_1})}{\sqrt{\gamma^2-\varepsilon_1}J_1(\min(j_{11}',R\sqrt{\varepsilon_2-\gamma^2}))[K_0(R\sqrt{\gamma^2-\varepsilon_1})+K_2(R\sqrt{\gamma^2-\varepsilon_1})]}\right|, \tag{5.1}$$

$$A_1(R) = 0.981460\times\left|\frac{K_1(\sqrt{R^2(\varepsilon_2-\varepsilon_1)-j_{01}^2})}{\{R\sqrt{R^2(\varepsilon_2-\varepsilon_1)-j_{01}^2}[K_0(\sqrt{R^2(\varepsilon_2-\varepsilon_1)-j_{01}^2})+K_2(\sqrt{R^2(\varepsilon_2-\varepsilon_1)-j_{01}^2})]\}}\right|, \quad 2.4\leqslant R\leqslant 5, \tag{5.2}$$

$$B_1(\alpha,R)=\frac{\min_{m=1,2}|g(\gamma_{1m}^2)|}{K_1(\sqrt{(\varepsilon_2-\varepsilon_1)R^2-j_{11}^2})0.3R^2\left(-\frac{\left[3\cos\left(\frac{1}{3}\cos^{-1}\left(\frac{\sqrt{|\alpha|}}{A_1}\right)-\frac{2\pi}{3}\right)\right]\|f\|_+}{\frac{\sqrt{|\alpha|}}{A_1}}\right)^3}, \tag{5.3}$$

with

$$\gamma_{1m}^2=\varepsilon_2-\frac{j_{1m}^2}{R^2},\quad m=1,2 \tag{5.4}$$

and

$$\|f\|_+=\left.\frac{k_1R|K_1'(k_1R)J_1(\min(j_{11}',k_2R))|}{k_2R|K_1(k_1R)J_1'(k_2R)|}\right|_{\gamma^2=\gamma_{12}^2}. \tag{5.5}$$

We choose $\varepsilon_1=1$, $\varepsilon_2=3.5$, and $\alpha=7\times10^{-4}$. In this case all sufficient conditions of theorems 1–4 in Ref. [10] are satisfied if $2.58<R<3.10$ (cf. Fig. 1). Figure 2 represents the solutions of the dispersion relation (4.2) for $\gamma^2\in\Lambda_1$. An associated field pattern is depicted in Fig. 3. The agreement of u_1 with the numerical solution (calculated by means of the NDSolve routine of Mathematica) is satisfactory [cf. Fig. 3(b)]. The intensity $E_0^2u^2(\rho\gamma^2)$ can be compared with experimental results.

The solution u_1 (for $\alpha_0=7\times10^{-4}$) can be continued analytically for $\alpha>\alpha_0$ (cf. Fig. 4). If $H(R,\gamma^{(0)},\alpha_0)=0$ and excluding $\partial H/\partial\gamma(R)|_{\alpha=\alpha_0}=0$ we obtain

$$\gamma=\gamma^{(0)}-\left.\frac{\frac{\partial H}{\partial\alpha}(R)}{\frac{\partial H}{\partial\gamma}(R)}\right|_{\alpha=\alpha_0}(\alpha-\alpha_0)+\cdots. \tag{5.6}$$

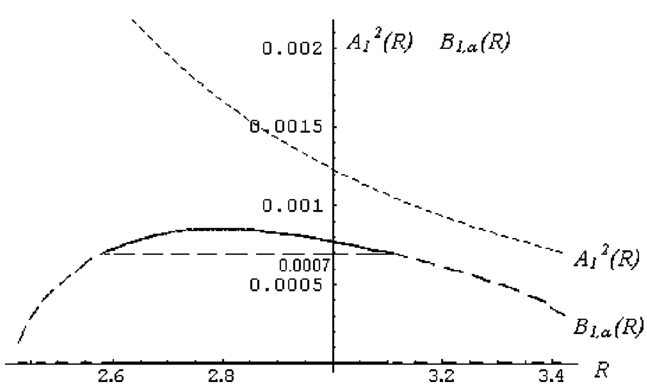

FIG. 1. Check of the sufficient conditions of theorem (cf. Appendix A) for $\alpha=7\times10^{-4}$. If $2.58<R<3.10$ holds, then $\alpha<\min\{A_1^2,B_{1,\alpha}\}$ is valid.

For $R=3.25$ and $\alpha_0=7\times10^{-4}$ the solution of the dispersion relation $H=0$ is $\gamma^{(0)}=1.597$. For $R=3.25$ and $\alpha=10^{-1}$ the solution is $\gamma_1=1.605$. Evaluation of Eq. (5.6) with $\alpha=10^{-1}$ leads to $\gamma_2=1.606$. The field patterns for this case are shown in Fig. 5. The agreement with the associated numerical solution is satisfactory (the singularity in the numerical solutions is due to the NDSolve routine of Mathematica). Hence it seems justified to conclude that the solution γ_1 of the dispersion relation (4.2) is a satisfactory approximate solution of the exact dispersion relation (4.1) though the (sufficient) conditions of the theorems 3 and 4 [10] are not satisfied. If we apply the same approach for $R=5$ and $\alpha=-10^{-1}$, we obtain the result shown in Fig. 6.

If a solution of the dispersion relation (4.2) exists in the vicinity of $\gamma=\sqrt{\varepsilon_1}$ $[R\to R_c=\sqrt{j_{11}^2/(\varepsilon_2-\varepsilon_1)}]$ the dependence of the cutoff radius R_c on the nonlinearity parameter α can be investigated in analogy to the linear case. In the limit $\gamma\to\sqrt{\varepsilon_1}$ Eq. (4.2) can be approximated by

$$kR_c\frac{J_0(kR_c)}{J_1(kR_c)}=\alpha\int_0^{R_c}r\ \frac{J_1(kr)}{J_1(kR_c)}^4dr, \tag{5.7}$$

where $k=\sqrt{\varepsilon_2-\varepsilon_1}$. Figure 7 shows the dependence of R_c on α.

The power flow P down the guide (in the core and in the cladding regime) is obtained from an integration of the z component of the time-averaged Poynting vector $\overline{S_Z}$ over the appropriate cross section:

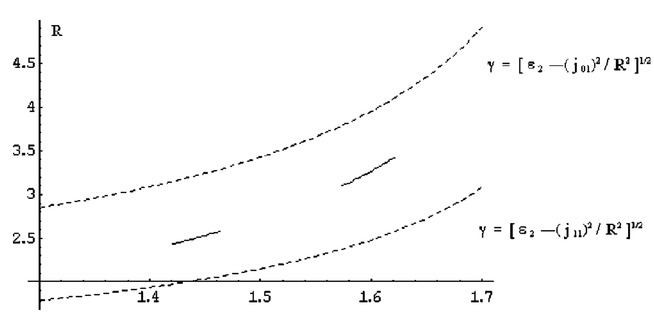

FIG. 2. Dependence of R on γ for $\alpha=7\times10^{-4}$. Dashed curves: boundaries of Λ_1; solid curves: solutions of the dispersion relation (4.2) subject to $\alpha<\min\{A_1^2,B_{1,\alpha}\}$ (cf. Fig. 1).

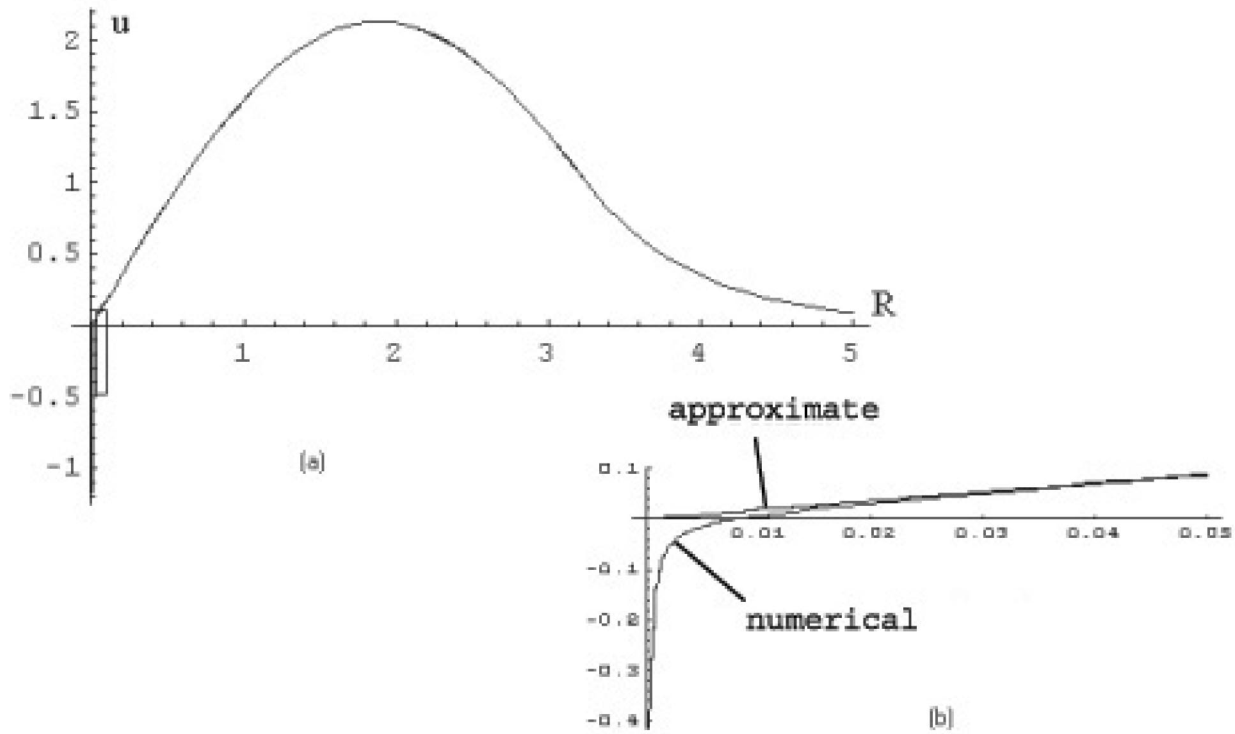

FIG. 3. (a) Field pattern according to Eqs. (3.2) and (4.3) (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$, $R=3.25$, $\alpha=7\times10^{-4}$) and numerical solution of Eqs. (2.9) and (2.10). (b) Enlarged fragment of the curve in the vicinity of origin.

$$P=P_{core}+P_{cl}, \tag{5.8}$$

with

$$P_{core}=2\pi\int_0^R d\rho\rho\overline{S_z}(k_2\rho), \tag{5.9}$$

$$P_{cl}=2\pi\int_R^\infty d\rho\rho\overline{S_z}(k_1\rho), \tag{5.10}$$

$$\overline{S_z}(s)=-\frac{1}{2}\,\mathrm{Re}\{E_\varphi(s)H_\rho^*(s)\}. \tag{5.11}$$

By using Eqs. (3.2) and (4.3) and asymptotic properties of $K_1(x)$ [14] Eqs. (5.9) and (5.10) can be evaluated to yield

$$P_{core}=2\pi\gamma P_0\int_0^R d\rho\rho u_1^2(k_2\rho), \tag{5.12}$$

$$P_{cl}=\pi R^2\gamma P_0\left(\frac{K_0(k_1R)K_2(k_1R)}{K_1^2(k_1R)}-1\right), \tag{5.13}$$

where $P_0=E_0^2k_0/2\omega\mu_0$. It is convenient to abbreviate $\beta=K_1^2(k_1R)/K_0(k_1R)K_2(k_1R)$. Thus the power flow fractions can be written as

$$\frac{P_{core}}{P}=\frac{2\beta\int_0^R dssu_1^2(k_2s)}{(1-\beta)R^2+2\beta\int_0^R dssu_1^2(k_2s)}, \tag{5.14}$$

$$\frac{P_{cl}}{P}=\frac{(1-\beta)R^2}{(1-\beta)R^2+2\beta\int_0^R dssu_1^2(k_2s)}. \tag{5.15}$$

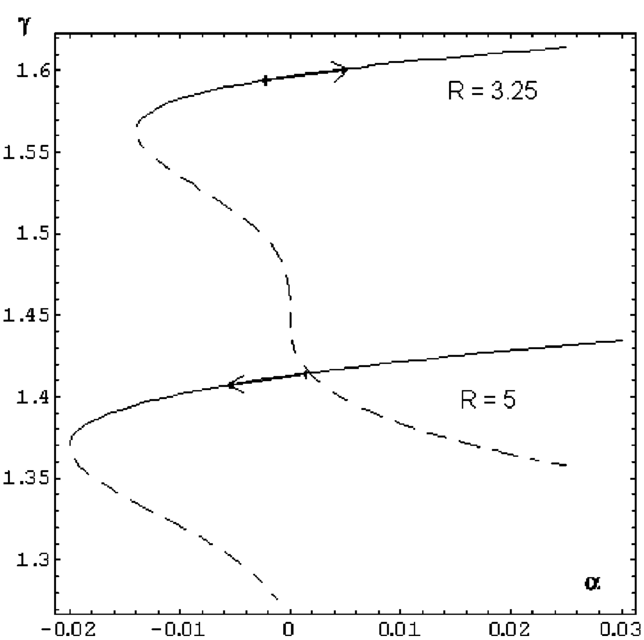

FIG. 4. Dependence of γ on α according to the dispersion relation (4.2) for $R=3.25$ and $R=5$. Arrows indicate analytic continuation (cf. details in the text). Dashing indicates nonphysical branches.

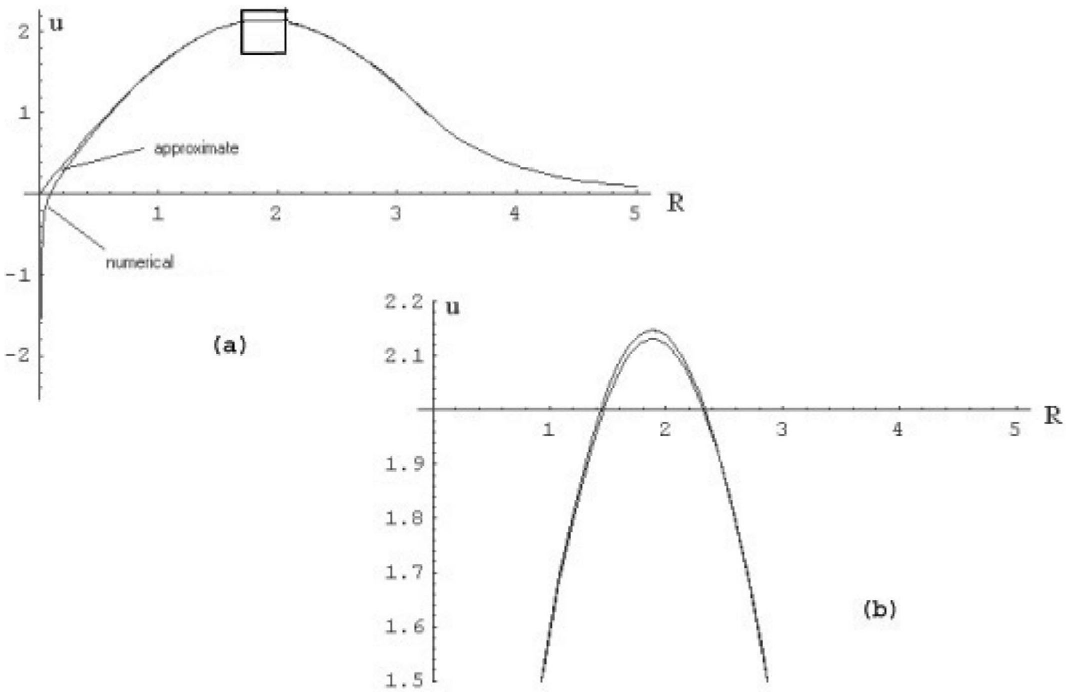

FIG. 5. (a) Field pattern according to Eqs. (3.2) and (4.3) (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$, $R=3.25$, $\alpha=10^{-1}$) and numerical solution of Eqs. (2.9) and (2.10). (b) Enlarged fragment of the curve in the vicinity of maximum.

Figure 8 shows the fraction $P_{core}/P=1-P_{cl}/P$ as a function of $\alpha=aE_0^2/\varepsilon_0$. Since an analytic continuation according to Eq. (5.6) for $\alpha\approx-0.06$ is possible it seems that a cutoff limit of α for defocusing material exists, where the power flow in the core switches to zero (we could not find associated experimental results in the literature). The dispersion curves related to Fig. 8 are shown in Fig. 9. Making use of Eqs. (5.12) and (5.13), the total power flow carried by the guided wave in first approximation is given by

$$P=\frac{\alpha\pi\varepsilon_0^{3/2}}{a\mu_0^{1/2}}\gamma\left(\int_0^R ds s u_1^2(k_2 s)+\frac{R^2(1-\beta)}{2\beta}\right). \quad (5.16)$$

According to the dispersion relation (4.2) the propagation constant γ is an implicit function of the radius R and of the nonlinearity parameter α. Thus, solving Eq. (4.2) for fixed α, one obtains the propagation constant γ as a function of the radius R: $\gamma=\gamma(R;\alpha=\text{const})$. By means of Eq. (5.14) we then find the corresponding values of the power flow P_{core}/P and

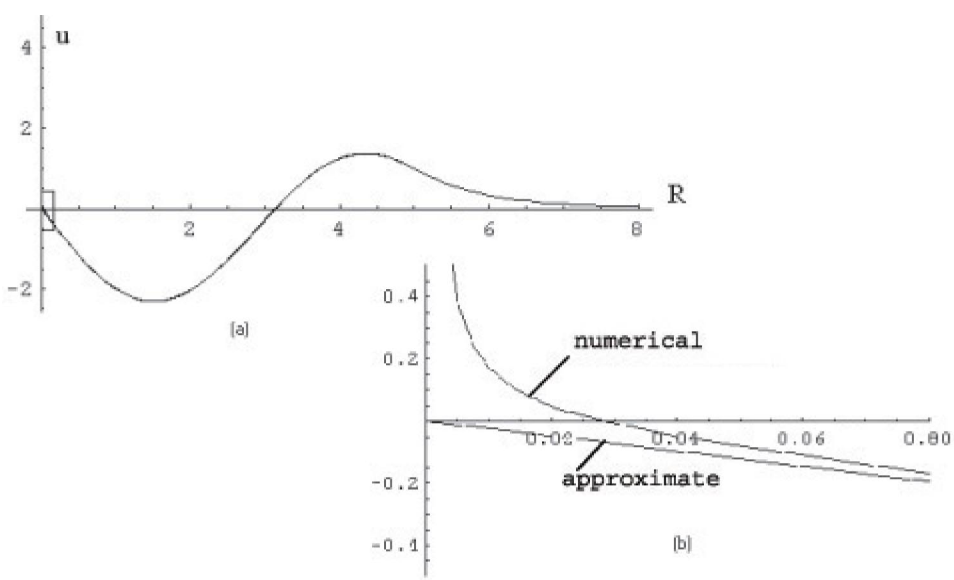

FIG. 6. Field pattern according to Eqs. (3.2) and (4.3) (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$, $R=5$, $\alpha=-10^{-1}$) and numerical solution of Eqs. (2.9) and (2.10). (b) Enlarged fragment of the curve in the vicinity of origin.

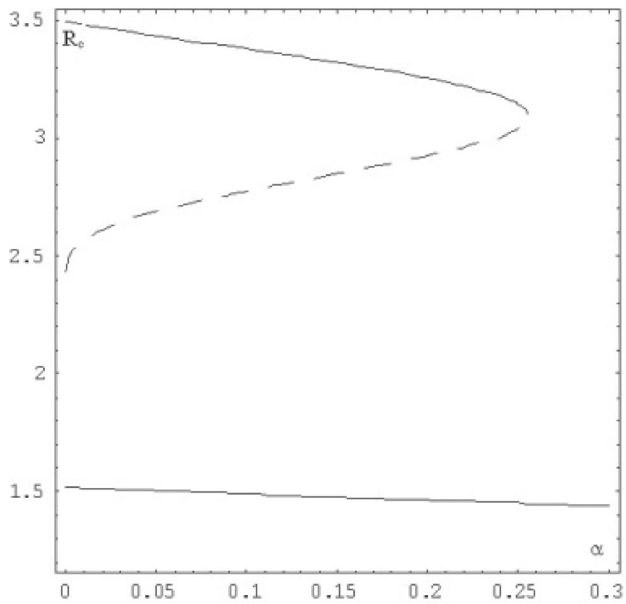

FIG. 7. Dependence of R_c on α (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$). Dashing indicates nonphysical branch.

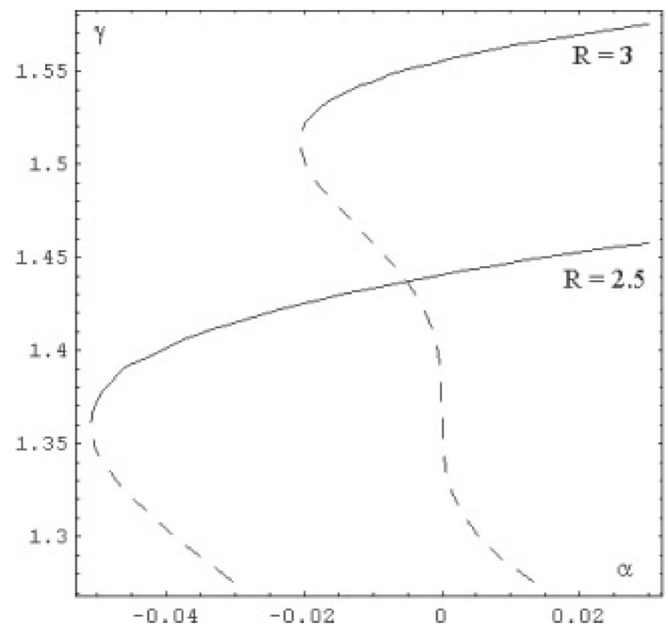

FIG. 9. Dependence of γ on α for $R=3$ and 2.5 (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$). Dashing indicates nonphysical branches.

its dependence on R. This dependence is shown in Fig. 10.

Finally, we evaluate the total power flow according to Eqs. (5.16). Apart from a constant (E_0 is assumed to be constant) the dependence of P on γ is given by the integral

$$I = \gamma \int_0^R ds s u^2(k_2 s) + \frac{R^2(1-\beta)}{\beta}, \tag{5.17}$$

with $\beta = K_1^2(k_1 R)/K_0(k_1 R) K_2(k_1 R)$ subject to the dispersion relation (4.2). We choose $\alpha = 0, \pm 0.01$. The associated dependence of R on γ is shown in Fig. 11.

The total power flow $P(\gamma)$ for selected branches of the dispersion relation (cf. Fig. 11) has been plotted in Fig. 12, where, in evaluating I, Eqs. (3.15) and (4.3) have been used for $\alpha=0$ and $\alpha=\pm 0.01$, respectively. Obviously the necessary condition for stability [21]

$$\frac{\partial P}{\partial \gamma} > 0 \tag{5.18}$$

is satisfied for the selected branches in Fig. 11.

We would like to make the following comment concerning the computational results: we solved the nonlinear Bessel equation numerically without approximation using the NDSolve Mathematica routine. The singularity at $\rho=0$ is generated by this routine and not due to $\alpha \neq 0$.

VI. CONCLUSION

The goal of this article has been to propose an approach to treat the propagation of electromagnetic (TE) waves in a cylindrical Kerr-nonlinear dielectric waveguide based on the Green's function method. We have obtained an approximate analytical solution of the nonlinear Bessel equation (2.10)

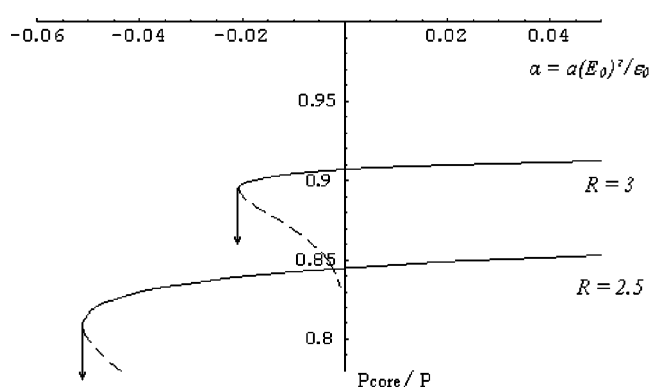

FIG. 8. Power flow fraction in the core with respect to $\alpha = aE_0^2/\varepsilon_0$ (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$). Arrows indicate switching of the power flow and dashing indicates nonphysical branches.

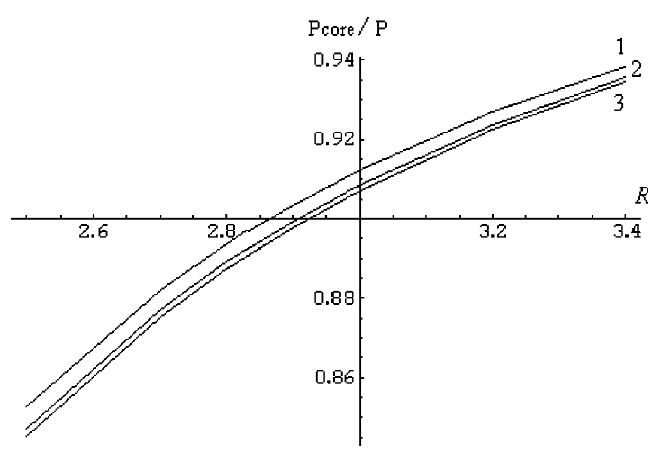

FIG. 10. Power flow P_{core}/P with respect to R (parameters: $\varepsilon_1=1$, $\varepsilon_2=3.5$); (1) $\alpha=0.05$, (2) $\alpha=0.01$, and (3) $\alpha=0.001$.

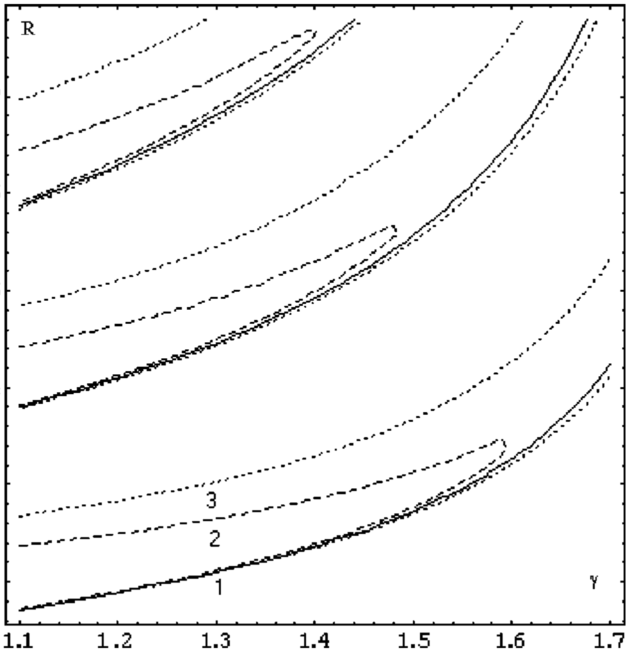

FIG. 11. Solutions $\{\gamma,R\}$ of the dispersion relation (4.2), in particular for $1.1\leqslant\gamma\leqslant1.44$ (cf. Fig. 12): (1) $\alpha=0$, solid curve; (2) $\alpha=-0.01$, dashed curve; (3) $\alpha=0.01$, dotted curve.

and an approximative solution of the dispersion relation (3.19). As indicated in Sec. V the approach is applicable to yield numerical results for field patterns, dispersion curves, cutoff radii, and power flow.

It seems that the approach can be applied to more general nonlinearities (e.g., higher order, saturating, photorefractive). Azimuthal polarization treated in this paper has experimental relevance [22]. We do not see how the approach can be used to investigate more general polarization cases, because, in place of Eqs. (2.7) and (2.10), in general we obtain two coupled nonlinear equations.

ACKNOWLEDGMENT

We thank Professor Valery S. Serov, University of Oulu, Finland, for insightful comments during a stay at Mathematisches Forschungsinstitut Oberwolfach, Germany.

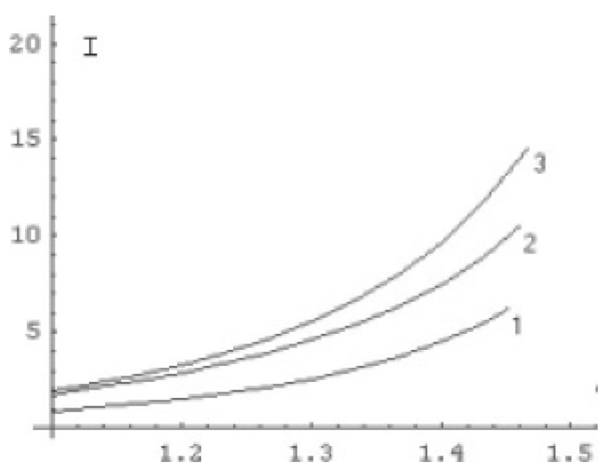

FIG. 12. Power flow integral I defined by Eq. (5.17) for particular solutions $\{\gamma,R\}$ of the dispersion relation (4.2) (cf. Fig. 11).

APPENDIX A

In Ref. [10] we proved theorem 3. As a consequence of the different normalization [compare Eqs. (3.2) and (2.21) in Ref. [10]] we rewrite theorem 3 in the following form.

Theorem. If ε_1, ε_2, and α satisfy the conditions $\varepsilon_2>\varepsilon_1>0$, and $0<|\alpha|<\alpha_0$, and $\varepsilon_2-j_{1m}^2/R^2>\varepsilon_1$ (for a certain $m\geqslant1$), where

$$\alpha_0=\min\{A_1^2(R),B_1(\alpha,R)\},\tag{A1}$$

$$A_1=\min_{\gamma^2\in\Lambda_1}A(R,\gamma^2),\tag{A2}$$

$$B_1=\frac{\min\limits_{m=1,2}|g(\gamma_{1m}^2)|}{0.3R^2\max\limits_{\gamma^2\in\Lambda_1}\{K_1(k_1R)r_-^3(\gamma^2)\}},\tag{A3}$$

$$r_-=-2\sqrt{\frac{1}{3\|N\|}}\cos\left(\frac{\arccos\left(\frac{3\sqrt{3}}{2}\|f\|\sqrt{\|N\|}\right)}{3}+\frac{2\pi}{3}\right),\tag{A4}$$

$$\|f\|=\max_{s\in[0,R]}|f(s)|=\max_{s\in[0,R]}\frac{k_1K_1'(k_1R)J_1(k_2s)}{k_2K_1(k_1R)J_1'(k_2R)},\tag{A5}$$

$$\|N\|=\max_{s\in[0,R]}\int_0^R|\alpha\rho G(\rho,s)|,\tag{A6}$$

$$\Lambda_1=\left[\varepsilon_2-\frac{j_{11}^2}{R^2},\varepsilon_2-\frac{j_{01}^2}{R^2}\right],\tag{A7}$$

[j_{0i} and j_{1i} denote the (positive) zeros of Bessel functions J_0, J_1] then at least one $\gamma^2\in\Lambda_1$ exists so that the problem described at the end of Sec. II has a nontrivial solution.

APPENDIX B: DERIVATION OF EQS. (5.1)–(5.3)

With respect to the definitions of A, A_1, and B_1 [cf. Eqs. (3.13), (A2), and (A3)] it is useful to estimate the quantity $T_G=\max_{s\in[0,R]}\int_0^R|\rho G(\rho,s)|ds$ first.

Introducing $t=k_2\rho$, $u=k_2s$, $U=k_2R$ and $h(u,U)=J_1(u)\times[N_1'(U)/J_1'(U)]-N_1(u)$ we obtain

$$\int_0^R|\rho G|d\rho=\frac{R}{2k^2}M(u,U),\tag{B1}$$

with $M(u,U)=|h(u,U)|\int_0^u dt\, t|J_1(t)|+|J_1(u)|\int_u^U dt\, t|h(t,U)|$. For simplicity we consider the first interval Λ_1 [cf. Eq. (A7)]. Thus, $j_{01}\leqslant U\leqslant j_{11}, 0\leqslant u\leqslant j_{11}$ are valid [which implies $J_1(u)\geqslant0$]. As shown in Fig. 13, $M(u,U)$ can be estimated from above according to $M(u,U)\leqslant M(j_{01},j_{01})$, so that

$$\max_{s\in[0,R]}\int_0^R|\rho G(\rho,s)|ds\leqslant\frac{\pi}{2k_2^2}|h(j_{01},j_{01})|\int_0^{j_{01}}dttJ_1(t).$$

By using $h(x,x)=2/\pi xJ_1'(x)$ [14] we finally obtain

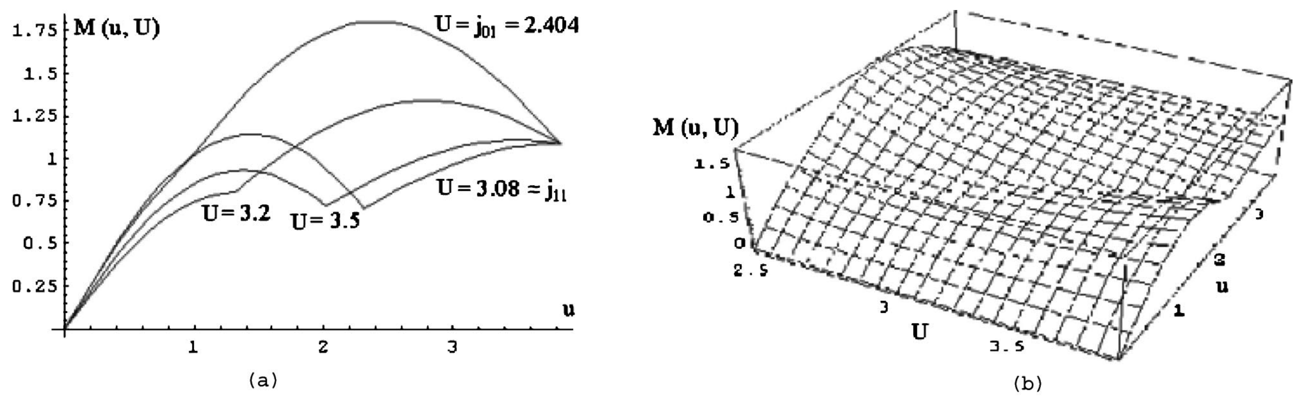

FIG. 13. Plots of the function $M(u,U)$ (cf. B_1): (a) for $0 \leq U \leq j_{11}$ and (b) for different u.

$$\max_{s\in[0,R]} \int_0^R |\rho G(\rho,s)| ds \leq \frac{\int_0^{j_{01}} dt\, t J_1(t)}{j_{01} J_1'(j_{01}) k_2^2} \approx \frac{2.832}{k_2^2}. \quad \text{(B2)}$$

Combining Eqs. (3.11), (3.13), and (B2) and using $K_1' = -\frac{1}{2}(K_0 + K_2)$ and $J_1' = \frac{1}{2}(J_0 - J_2)$ one obtains Eq. (5.1), since $\max_{s\in[0,R]} J_1(k_2 s) = J_1(\min\{j_{11}', k_2 R\})$ holds $[j_{11}' \approx 1.841$ denotes the first zero of $J_1'(x)]$.

A plot of $A(\gamma^2, R)$ (e.g., for $\varepsilon_1 = 1$, $\varepsilon_2 = 3.5$, $2.4 \leq R \leq 5$, and $\gamma^2 \in \Lambda_1$) shows that the minimum of A with respect to $\gamma^2 \in \Lambda$ is given by $A(\varepsilon_2 - j_{01}^2/R^2, R)$. Evaluation yields Eq. (5.2).

To derive Eq. (5.3) we combine Eqs. (A4)–(A6) to obtain

$$r_- = -\frac{2}{\sqrt{3|\alpha| T_G}} \cos\left\{\frac{\arccos\left(\frac{3\sqrt{3}}{2}\|f\|\sqrt{|\alpha| T_G}\right)}{3} - \frac{2\pi}{3}\right\}, \quad \text{(B3)}$$

or, taking into account Eq. (3.13),

$$r_- = \|f\| \delta\left(\frac{\sqrt{|\alpha|}}{A}\right), \quad \text{(B4)}$$

with

$$\delta(x) = -\frac{3}{x} \cos\left(\frac{\arccos x}{3} - \frac{2\pi}{3}\right). \quad \text{(B5)}$$

Function $\delta(x)$ is monotonically increasing. $K_1(k_1 R)$ in Eq. (A3) is a monotonically decreasing function and $\|f\|$ a monotonically increasing function with respect to $\gamma^2 \in \Lambda$. Thus $B_1(\alpha, R)$ defined by Eq. (A3) can be evaluated according to Eq. (5.3).

[1] F. Borgnis and C. Papas, in *Electromagnetic Waveguides*, Handbuch der Physik Vol. 16, edited by S. Flügge (Springer-Verlag, Berlin, 1958), p. 24; P. J. B. Clarricoats, in *Progress in Optics XIV*, edited by E. Wolf (North-Holland, Amsterdam, 1976).
[2] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[3] R. Y. Chiao *et al.*, Phys. Rev. Lett. **13**, 479 (1964).
[4] V. M. Eleonski *et al.*, Sov. Phys. JETP **35**, 44 (1972).
[5] Y. Chen, J. Opt. Soc. Am. B **8**, 2338 (1991).
[6] N. N. Akhmediev, V. I. Korneev, and R. F. Nabiev, Phys. Rev. A **46**, 430 (1992).
[7] R. A. Sammut and C. Pask, J. Opt. Soc. Am. B **8**, 395 (1990).
[8] D. Sjöberg, Radio Sci. **38**, 395 (2003).
[9] H. W. Schürmann, V. S. Serov, and Y. Shestopalov, Phys. Rev. E **58**, 1040 (1998).
[10] Y. Smirnov, H. W. Schürmann, and Y. Shestopalov, J. Nonlinear Math. Phys. **11**, 256 (2004).
[11] Cf. Ref. [10], Theorem 5.
[12] Cf. Ref. [10], Theorems 3 and 4.
[13] I. Stakgold, *Green's Functions and Boundary Value Problems* (Wiley, New York, 1998), p. 221.
[14] A. Snyder and J. Love, *Optical Waveguide Theory* (Chapman and Hall, London, 1983), p. 714.
[15] Cf. Ref. [10], Theorem 1.
[16] Cf. Ref. [10], Theorem 2.
[17] Cf. Ref. [10], Propos. 1.
[18] D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), p. 367.
[19] G. Agrawal, *Nonlinear Fiber Optics* (Academic, New York, 2001), Chap. 2.3.
[20] Cf. Ref. [10], Theorem 4, Theorem 5.
[21] N. N. Akhmediev and A. Ackiewicz, *Solitons, Nonlinear Pulses and Beams* (Chapman and Hall, London, 1997), pp. 236–238.
[22] See K. S. Youngworth and T. G. Brown, Opt. Express **7**, 77 (2000); R. Dorn, S. Quabis, and G. Leuchs, Phys. Rev. Lett. **91**, 233901 (2003).